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TABLES OF GREEN'S FUNCTIONS FOR THE THEORY OF BEAM VIBRATIONS WITH GENERAL INTERMEDIATE APPENDAGES

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Abstract—The paper deals with the construction and tabulation of Green's functions in a form suitable for use in determining natural frequencies and mode shapes of beams with intermediate attachments and of various boundary conditions. The beam may have rotational and linear elastic attachments, as well as rotational and linear attached inertias. Example computations are illustrated.

NOMENCLATURE

ω	radian frequency
m	beam mass/unit length
EI	beam flexural rigidity
L	beam length
q	$q^4 = \omega^2 m/EI$
z	$z = qL$
x, u	arguments of Green's function
$\bar{x}$, $\bar{u}$	x/L, u/L, respectively
c, C, s, S	cos z, cosh z, sin z, sinh z, respectively
$\hat{c}, \hat{C}, \hat{s}, \hat{S}$	$\cos z(1-\bar{u})$, $\cosh z(1-\bar{u})$, $\sin z(1-\bar{u})$, $\sinh z(1-\bar{u})$, respectively
$\tilde{c}, \tilde{C}, \tilde{s}, \tilde{S}$	cos $z\bar{x}$, cosh $z\bar{x}$, sin $z\bar{x}$, sinh $z\bar{x}$, respectively
c^*, C^*, s^*, S^*	cos $z\bar{u}$, cosh $z\bar{u}$, sin $z\bar{u}$, sinh $z\bar{u}$, respectively
a_i	location of ith translational spring
k_{Ti}	ith translational spring
b_i	location of ith attached mass
m_i	mass of ith disc
I_{Pi}	diameteral mass moment of inertia of ith disc
k_{Ri}	ith rotational spring
c_i	location of ith attached rotational spring
$(\ )'$	total derivative with respect to x
$(\ )_{,\zeta}$	partial derivative with respect to ζ
Δ_B	determinant of the matrix $[B]$
$Y(x)$	constrained beam mode shape
$\{Y\}$	$[Y(a_1)\ldots Y(a_N)Y(b_1)\ldots Y(b_r)LY'(b_1)\ldots LY'(b_r)\ldots LY'(c_1)\ldots LY'(c_n)]^T$.

1. INTRODUCTION

Beam models with intermediate inertia and elastic attachments are a common occurrence in practice. These attachments alter the dynamic characteristics of the base beam in manners described by Strut and Rayleigh (1945). The effect of a mass attachment on the natural frequencies of a beam with several boundary conditions was studied by Maltbaek (1961), where two displacement functions were assumed for the two segments of the beam. By application of the boundary and continuity requirements, the frequency equation appeared. By using a Lagrangian formulation, Dowell (1979) derived the equation for the natural frequencies of a combined system consisting of a simply-supported beam and an oscillator (spring-mass) attached at some intermediate point on the beam. Using the direct method stated in Maltbaek (1961), Lau (1984) derived and solved the frequency equation for a cantilever beam with one linear and one rotational spring attached to the same point on the cantilever. With more than one attachment at different points on the beam, the above techniques become unwieldy, and approximate methods such as in Verniere de Irassar *et al.* (1984), or the Green's function formulation of Nicholson and Bergman (1986) may now be applied. The use of the Green's functions not only leads to exact results, but also involves less cumbersome manipulations. The use of this method depends on the availability of the Green's function corresponding to the beam with associated boundary conditions.

In this paper these functions as well as their relevant derivatives are determined and tabulated for beams with general non-classical boundary conditions covering a wide range of practical problems. These functions are presented in a separable form, where the separate functional components are evaluated from 2×2 matrices.

2. THE EQUATION OF MOTION

The beam, with a segment shown in Fig. 1, has N linear springs, n rotational springs, and r masses with both linear and rotational inertias. The exact boundary conditions need not be specified yet.

The lateral displacement $y(x, t)$ of the beam is governed by the differential equation:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+m\frac{\partial^2 y(x,t)}{\partial t^2} = -\sum_{i=1}^{N} k_{Ti}y(a_i,t)\delta(x-a_i) - \sum_{j=1}^{r} m_j\frac{\partial^2 y(b_j,t)}{\partial t^2}\delta(x-b_j)+\sum_{j=1}^{r} I_{Pj}\frac{\partial^2}{\partial t^2}\left\{\frac{\partial y(b_j,t)}{\partial x}\right\}\delta'(x-b_j) + \sum_{i=1}^{n} k_{Ri}\frac{\partial y(c_i,t)}{\partial x}\delta'(x-c_i), \quad (1)$$

where $\delta(x-a)$ is the Dirac delta function whose properties used in the text are, from Bracewell (1978),

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a), \quad (2a)$$

$$\int_{-\infty}^{+\infty} f(x)\delta'(x-a)\,\mathrm{d}x = -f'(a). \quad (2b)$$

The standard separation of variables assumption applies to the solution, thus

$$y(x,t) = Y(x)\,\mathrm{e}^{\mathrm{i}\omega t}. \quad (3)$$

Furthermore if $G(x, u)$ is the Green's function for the stated problem with specified boundary conditions, then the solution to (1) with (2a), (2b) and (3) utilized is

$$EIY(x) = -\sum_{i=1}^{N} k_{Ti}G(x,a_i)Y(a_i)-\sum_{i=1}^{n} k_{Ri}G_u(x,c_i)Y'(c_i) + \sum_{j=1}^{r} \omega^2 m_j G(x,b_j)Y(b_j)+\sum_{j=1}^{r} \omega^2 I_{P_j}G_u(x,b_j)Y'(b_j). \quad (4)$$

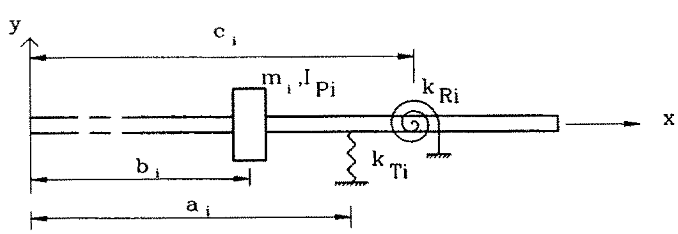


Fig. 1. Beam segment with general attachments.

The Green's function and its relevant derivatives can be given in the forms:

$$G(x,u) = \frac{1}{2\Delta_e q^3}\begin{cases} g(x,u), & 0 \leqslant x \leqslant u, \\ g(u,x), & L \geqslant x \geqslant u, \end{cases} \tag{5a}$$

$$G_u(x,u) = \frac{1}{2\Delta_e q^2}\begin{cases} f(x,u), & 0 \leqslant x \leqslant u, \\ v(u,x), & L \geqslant x \geqslant u, \end{cases} \tag{5b}$$

$$G_x(x,u) = \frac{1}{2\Delta_e q^2}\begin{cases} v(x,u), & 0 \leqslant x \leqslant u, \\ f(u,x), & L \geqslant x \geqslant u, \end{cases} \tag{5c}$$

$$G_{xu}(x,u) = \frac{1}{2\Delta_e q}\begin{cases} h(x,u), & 0 \leqslant x \leqslant u, \\ h(u,x), & L \geqslant x \geqslant u, \end{cases} \tag{5d}$$

where Δ_e and all functions on the right-hand sides are still to be determined. If (5a), (5b), (5c) and (5d) are substituted into (4) and its derivative, the resulting equations in dimensionless attachment parameters are, assuming $0 \leqslant x \leqslant u$,

$$2\Delta_e z^3 Y(x) = -\sum_{i=1}^{N} K_i g(x,a_i) Y(a_i) + \sum_{i=1}^{r} z^4 M_i g(x,b_i) Y(b_i) + \sum_{i=1}^{r} z^5 J_i f(x,b_i)[LY'(b_i)] - \sum_{i=1}^{n} zQ_i f(x,c_i)[LY'(c_i)], \tag{6}$$

$$2\Delta_e z^2[LY'(x)] = -\sum_{i=1}^{N} K_i v(x,a_i) Y(a_i) + \sum_{i=1}^{r} z^4 M_i v(x,b_i) Y(b_i) + \sum_{i=1}^{r} z^5 J_i h(x,b_i)[LY'(b_i)] - \sum_{i=1}^{n} zQ_i h(x,c_i)[LY'(c_i)], \tag{7}$$

where

$$K_i = k_{Ti}L^3/EI, \quad Q_i = k_{Ri}L/EI, \quad M_i = m_i/mL, \quad J_i = I_{Pi}/mL^3.$$

3. FREQUENCIES AND MODE SHAPES

The frequencies are determined from eqns (6) and (7). Equation (6) is evaluated at all $N+r$ points of linear spring and linear inertia attachments, giving $N+r$ equations. Equation (7) is then evaluated at all $n+r$ points of rotational attachments, giving $n+r$ equations. The total of $N+n+2r$ equations is now assembled in the matrix form

$$[D]\{Y\} = \{0\}. \tag{8}$$

The requirement of a non-trivial solution of (8) yields the frequency equation

$$\Delta_D = 0 \tag{9}$$

from which the roots z_j (frequency parameters) are found, and hence the frequencies

$$\omega_j = z_j^2\sqrt{EI/mL^4}. \tag{10}$$

The mode shapes corresponding to a given frequency z_j can be obtained only to within a constant. Thus one may specify $Y_j(a_1)$ for example, and use the first $N+n+2r-1$ equations of (8) to find the remaining elements of $\{Y\}$. Letting

$$A_{ji} = Y_j(a_i)/Y_j(a_1), \qquad i = 2,\ldots,N, \tag{11a}$$

$$B_{ji} = Y_j(b_i)/Y_j(a_1), \qquad i = 1,\ldots,r, \tag{11b}$$

$$D_{ji} = LY'_j(b_i)/Y_j(a_1), \quad i = 1, \ldots, r, \tag{11c}$$

$$C_{ji} = LY'_j(c_i)/Y_j(a_1), \quad i = 1, \ldots, n, \tag{11d}$$

the mode shapes from (6) are:

$$U_j(x) = -K_1 g_j(x, a_1) - \sum_{i=2}^{N} A_{ji} K_i g_j(x, a_i) + \sum_{i=1}^{r} z_j^4 B_{ji} M_i g_j(x, b_i)$$
$$+ \sum_{i=1}^{r} z_j^5 D_{ji} J_i f_j(x, b_i) - \sum_{i=1}^{n} z_j C_{ji} Q_i f_j(x, c_i), \tag{12}$$

where $U_j(x) = 2\Delta_{ej} z_j^3 Y_j(x)/Y_j(a_1)$.

This completes the generalized formulae for the natural frequencies and mode shapes via (9) and (12).

4. DETERMINATION OF THE GREEN'S FUNCTIONS

The Green's functions for beams of generalized boundary conditions are now evaluated for use with the standardized procedure above. The functions are assumed in the form:

$$G(x, u) = \begin{cases} D_1 \cos qx + D_2 \sin qx + D_3 \cosh qx + D_4 \sinh qx, & 0 \leqslant x \leqslant u, \\ D_5 \cos qx + D_6 \sin qx + D_7 \cosh qx + D_8 \sinh qx, & L \geqslant x \geqslant u, \end{cases} \tag{13}$$

where $G(x, u)$ must:

(a) Satisfy two boundary conditions at each end. (Shear force and bending moment are positive and negative respectively at $x = 0$, but negative and positive respectively at $x = L$.)

(b) Fulfill displacement, slope and moment continuity at $x = u$, i.e.

$$G(u^+, u) = G(u^-, u), \tag{14a}$$

$$G_x(u^+, u) = G_x(u^-, u), \tag{14b}$$

$$G_{xx}(u^+, u) = G_{xx}(u^-, u). \tag{14c}$$

(c) Satisfy a shear force discontinuity of magnitude EI at $x = u$, i.e.

$$EI[G_{xxx}(u^+, u) - G_{xxx}(u^-, u)] = EI. \tag{15}$$

Using (14a)–(15), D_5–D_8 in (13) can be expressed in terms of D_1–D_4, thus

$$D_5 = D_1 + as^*, \quad D_6 = D_2 - ac^*, \quad D_7 = D_3 - aS^*, \quad D_8 = D_4 + aC^*, \tag{16}$$

where $a = 1/2q^3$.

Equations (16) which are true for all boundary conditions, are now used with (a) to establish the Green's function for a beam with any specified boundary conditions.

5. PRESENTATION OF THE GREEN'S FUNCTIONS

The Green's functions as determined by the above procedure are now presented in the most suitable form for numerical computations. As a result, $g(x, u)$, $f(x, u)$, $v(x, u)$ and $h(x, u)$ in (5) are determined in a separable form for $u \geqslant x \geqslant 0$, from

$$g(x, u) = \psi_{11}(u)\phi_{11}(x) + \psi_{21}(u)\phi_{21}(x), \tag{17a}$$

$$f(x, u) = \psi_{12}(u)\phi_{11}(x) + \psi_{22}(u)\phi_{21}(x), \tag{17b}$$

$$v(x,u) = \psi_{11}(u)\phi_{12}(x)+\psi_{21}(u)\phi_{22}(x), \tag{17c}$$

$$h(x,u) = \psi_{12}(u)\phi_{12}(x)+\psi_{22}(u)\phi_{22}(x). \tag{17d}$$

Elements of the $[\psi]$ matrix are given in terms of elements of the two matrices $[e]$ and $[\varphi]$, which are in turn given in Table 5, thus

$$\psi_{11}(u) = \varphi_{11}(u)e_{22}-\varphi_{21}(u)e_{12}, \tag{18a}$$

$$\psi_{12}(u) = \varphi_{12}(u)e_{22}-\varphi_{22}(u)e_{12}, \tag{18b}$$

$$\psi_{21}(u) = \varphi_{21}(u)e_{11}-\varphi_{11}(u)e_{21}, \tag{18c}$$

$$\psi_{22}(u) = \varphi_{22}(u)e_{11}-\varphi_{12}(u)e_{21}. \tag{18d}$$

In order to minimize duplications in presenting the Tables, note is taken of the facts that the $[\varphi]$ matrix contains only boundary information at $x = L$. Beams of the same conditions at $x = L$ have the same $[\varphi]$ and Δ_φ. Similarly, beams of the same end conditions at $x = 0$ have the same $[\phi]$ and Δ_ϕ. The $[e]$ matrix is constructed from boundary information at both ends, and is different for all cases. $\Delta_e = 0$ is the frequency equation of the beam with the specified boundary conditions but with no intermediate attachments. Finally, the symmetry of the Green's functions can be expressed as $g(x,u) = g(u,x)$, $f(x,u) = v(u,x)$, and $h(x,u) = h(u,x)$. These functions are therefore given only for $0 \leqslant x \leqslant u$.

6. A SPECIAL CASE

For a beam with only two intermediate elements, one linear and one rotational both attached at $x = a$, expanding the corresponding 2×2 determinant of (9) gives rise to

$$\Delta_e^2+\Delta_e\left[\frac{K}{2z^3}g(a,a)+\frac{Q}{2z}h(a,a)\right]+\frac{KQ}{4z^4}[g(a,a)h(a,a)-f(a,a)v(a,a)] = 0, \tag{19}$$

where a linear spring K and a rotational spring Q are considered for demonstration. The difference in square brackets of (19) can be shown to be

$$g(a,a)h(a,a)-f(a,a)v(a,a) = \Delta_e\Delta_\phi\Delta_\varphi. \tag{20}$$

Hence each term of (19) contains the factor Δ_e. Since $\Delta_e = 0$ is the frequency equation of the beam without attachments, an attempt to solve (19) as it stands will give mixed results for the stated problem as well as for $\Delta_e = 0$. Using (20), the desired form of (19) is

$$\Delta_e+\left[\frac{K}{2z^3}g(a,a)+\frac{Q}{2z}h(a,a)\right]+\frac{KQ}{4z^4}\Delta_\phi\Delta_\varphi = 0. \tag{21}$$

7. EXAMPLE PROBLEMS

The formulae in Table 5 are verified as far as possible using published data where available. For a fixed–fixed beam with one intermediate linear spring attachment, the frequency equation from (9) is

$$2z^3\Delta_e+Kg(a,a) = 0. \tag{22}$$

Using Δ_e and $g(a,a)$ from Table 5(b), the fundamental frequency coefficients (z_1^2) resulting from the solution of (22) are given in Table 1, and compared with results from Verniere de Irassar *et al.* (1984).

Applying (21) to a constrained cantilever with a free tip, Table 5(c) is used, and the frequencies for the first five modes are given for $a/L = 0.8$ and various combinations of K

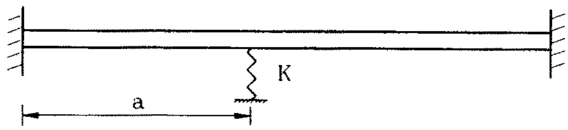
and Q. These results are shown in Table 2 and agree perfectly with Lau (1984). For a cantilever with an attached disc which has both linear and rotational inertias, the frequency equation, after applying (20) and cancellation of the common terms Δ_e, is

$$4\Delta_e - 2[zMg(b,b) + z^3Jh(b,b)] + z^4MJ\Delta_\phi\Delta_\varphi = 0. \tag{23}$$

The solution of (23) for $b/L = 0.4$ with various values of M and J is given in Table 3 for the first five modes ($z_n, n = 1, \ldots, 5$).

The final example is a cantilever with tip inertias and elasticities, with an intermediate linear spring and mass with both linear and rotational inertias. The frequency determinant

Table 1. Frequency coefficients (z_n^2) for a fixed–fixed beam with an intermediate spring [values in brackets from Verniere de Irassar *et al.* (1984)]



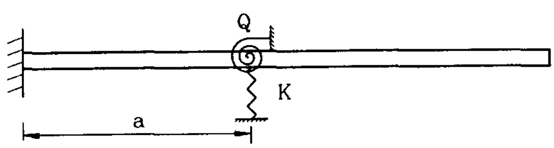
		K			
Mode	a/L	1.00	10.00	50.00	100.00
1	0.0	22.37326	22.37326	22.37326	22.37326
		(22.43)	(22.43)	(22.43)	(22.43)
	0.1	22.37407	22.38124	22.41283	22.45163
		(22.44)	(22.44)	(22.48)	(22.53)
	0.2	22.38184	22.45828	22.78436	23.16301
		(22.44)	(22.53)	(22.90)	(23.34)
	0.3	22.40009	22.63808	23.62914	24.73352
		(22.46)	(22.71)	(23.77)	(24.97)
	0.4	22.42055	22.83983	24.58139	26.52217
		(22.48)	(22.90)	(24.66)	(26.64)
	0.5	22.42955	22.92906	25.01149	27.35681
		(22.49)	(22.98)	(25.03)	(27.38)
2	0.0	61.67281	61.67281	61.67281	61.67281
	0.1	61.67450	61.68962	61.75644	61.83893
	0.2	61.68460	61.79057	62.25562	62.82330
	0.3	61.69118	61.85675	62.59583	63.52522
	0.4	61.68149	61.75998	62.11668	62.58049
	0.5	61.67281	61.67281	61.67281	61.67281
3	0.0	120.90338	120.90338	120.90338	120.90338
	0.1	120.90581	120.92787	121.02547	121.14647
	0.2	120.91280	120.99744	121.37431	121.84669
	0.3	120.90649	120.93465	121.06120	121.22276
	0.4	120.90501	120.91974	120.98564	121.06909
	0.5	120.91156	120.98523	121.31430	121.72943
4	0.0	199.85944	199.85944	199.85944	199.85944
	0.1	199.86232	199.88831	200.00352	200.14694
	0.2	199.86380	199.90300	200.07828	200.29948
	0.3	199.85989	199.86391	199.88187	199.90440
	0.4	199.86429	199.90805	200.10313	200.34836
	0.5	199.85944	199.85944	199.85944	199.85944
5	0.0	298.55493	298.55493	298.55493	298.55493
	0.1	298.55811	298.58447	298.70117	298.84717
	0.2	298.55615	298.56299	298.59351	298.63135
	0.3	298.55811	298.58496	298.70532	298.85620
	0.4	298.55493	298.55615	298.55933	298.56348
	0.5	298.55859	298.58862	298.72314	298.89160

from (9) is

$$\begin{vmatrix} 2\Delta_e z^3 + Kg(a,a) & -z^4 Mg(a,b) & -z^5 Jf(a,b) \\ Kg(b,a) & 2\Delta_e z^3 - z^4 Mg(b,b) & -z^5 Jf(b,b) \\ Kv(b,a) & -z^4 Mv(b,b) & 2\Delta_e z^2 - z^5 Jh(b,b) \end{vmatrix} = 0. \quad (24)$$

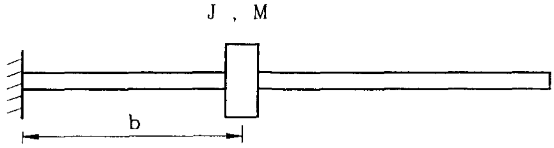
Using the symmetries that $g(a,b) = g(b,a)$ and $f(a,b) = v(b,a)$, the first five modes ($z_n, n = 1, \ldots, 5$), are given in Table 4, after the relevant Green's functions were constructed from Table 5(c).

Table 2. Frequency parameters (z_n) for a cantilever with linear and rotational springs ($a/L = 0.8$)



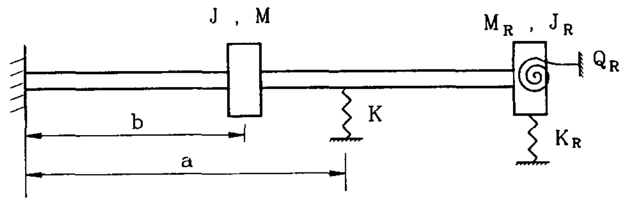
	Q					
K	0	1	10	100	1000	10000
			Mode 1			
0	1.87510	2.07404	2.45029	2.59156	2.60974	2.61161
1	1.95026	2.13046	2.48409	2.61940	2.63687	2.63867
10	2.40287	2.50532	2.73864	2.83611	2.84894	2.85026
100	3.82712	3.84606	3.88307	3.89618	3.89781	3.89798
1000	4.67897	4.81700	5.30157	5.54922	5.57823	5.58115
10000	4.68231	4.82670	5.40219	5.81881	5.87973	5.88605
			Mode 2			
0	4.69409	4.85644	5.59037	6.25502	6.36381	6.37514
1	4.69414	4.85655	5.59075	6.25560	6.36442	6.37575
10	4.69458	4.85753	5.59419	6.26091	6.36996	6.38133
100	4.70206	4.87246	5.63556	6.31920	6.43036	6.44193
1000	6.20838	6.25651	6.62687	7.20711	7.30972	7.32023
10000	7.15258	7.21528	7.68878	8.78617	9.07182	9.09899
			Mode 3			
0	7.85476	7.90050	8.25447	9.24241	9.60612	9.65050
1	7.85508	7.90080	8.25466	9.24245	9.60614	9.65051
10	7.85798	7.90355	8.25644	9.24284	9.60629	9.65063
100	7.88786	7.93186	8.27459	9.24680	9.60775	9.65186
1000	8.24681	8.27298	8.49487	9.28862	9.62208	9.66372
10000	9.46539	9.46705	9.48157	9.58481	9.71341	9.73721
			Mode 4			
0	10.99554	10.99802	11.01764	11.09928	11.15645	11.16557
1	10.99585	10.99833	11.01796	11.09963	11.15682	11.16594
10	10.99865	11.00114	11.02086	11.10280	11.16010	11.16924
100	11.02691	11.02951	11.05003	11.13467	11.19318	11.20246
1000	11.32576	11.32949	11.35845	11.49620	11.53850	11.54902
10000	12.86380	12.87324	12.94350	13.16842	13.28256	13.29856
			Mode 5			
0	14.13717	14.14303	14.18541	14.30839	14.36520	14.37294
1	14.13729	14.14316	14.18553	14.30849	14.36530	14.37304
10	14.13844	14.14430	14.18661	14.30941	14.36614	14.37387
100	14.15006	14.15584	14.19758	14.31869	14.37465	14.38227
1000	14.27810	14.28303	14.31861	14.42158	14.46915	14.47563
10000	16.02277	16.02319	16.02606	16.03304	16.03580	16.03616

Table 3. Frequency parameters (z_n) for a cantilever with an intermediate mass with rotational inertia ($b/L = 0.4$)



		J					
Mode	M	0.0	0.2	0.4	0.6	0.8	1.0
1	0.0	1.87510	1.59516	1.44037	1.33875	1.26484	1.20754
	0.2	1.85560	1.58633	1.43514	1.33516	1.26215	1.20542
	0.4	1.83672	1.57765	1.42996	1.33160	1.25948	1.20331
	0.6	1.81846	1.56913	1.42484	1.32806	1.25683	1.20121
	0.8	1.80082	1.56075	1.41978	1.32455	1.25419	1.19912
	1.0	1.78378	1.55253	1.41477	1.32107	1.25157	1.19705
2	0.0	4.69409	3.46942	3.24743	3.16212	3.11708	3.08925
	0.2	4.34004	3.46916	3.24324	3.15544	3.10889	3.08008
	0.4	4.10057	3.46881	3.23808	3.14746	3.09925	3.06936
	0.6	3.92678	3.46831	3.23166	3.13788	3.08787	3.05682
	0.8	3.79409	3.46754	3.22354	3.12632	3.07442	3.04217
	1.0	3.68900	3.46621	3.21311	3.11236	3.05858	3.02516
3	0.0	7.85476	4.91798	4.89924	4.89353	4.89076	4.88914
	0.2	7.57369	4.48567	4.46611	4.46054	4.45791	4.45638
	0.4	7.44313	4.20018	4.18058	4.17548	4.17314	4.17180
	0.6	7.36903	3.99419	3.97577	3.97153	3.96966	3.96860
	0.8	7.32156	3.83675	3.82111	3.81807	3.81678	3.81606
	1.0	7.28861	3.71184	3.70088	3.69919	3.69852	3.69815
4	0.0	10.99554	8.56665	8.56039	8.55832	8.55729	8.55666
	0.2	10.85163	8.28697	8.27989	8.27754	8.27636	8.27566
	0.4	10.78372	8.16018	8.15267	8.15014	8.14892	8.14819
	0.6	10.74503	8.08893	8.08114	8.07859	8.07725	8.07651
	0.8	10.72019	8.04354	8.03554	8.03288	8.03159	8.03081
	1.0	10.70293	8.01209	8.00401	8.00128	7.99996	7.99916
5	0.0	14.13717	12.39830	12.39337	12.39170	12.39087	12.39038
	0.2	13.41916	12.22389	12.21916	12.21759	12.21683	12.21638
	0.4	13.17939	12.12240	12.11788	12.11648	12.11573	12.11528
	0.6	13.06275	12.06067	12.05640	12.05496	12.05420	12.05380
	0.8	12.99424	12.02009	12.01585	12.01445	12.01374	12.01332
	1.0	12.94930	11.99154	11.98733	11.98598	11.98529	11.98487

Table 4. Frequency parameters (z_n) for a cantilever with an intermediate mass and spring, with a loaded tip. $J_R = 0.2$, $M_R = 0.2$, $K_R = 200$, $Q_R = 100$, $a/L = 0.6$, $b/L = 0.4$, $M = 0.2$



		K					
Mode	J	0	1	10	100	1000	10000
1	0.0	3.85459	3.86047	3.91189	4.31514	4.77617	4.77644
	0.2	2.75779	2.76039	2.78267	2.93176	3.17387	3.23470
	0.4	2.35193	2.35387	2.37052	2.48204	2.67260	2.72574
	0.6	2.13444	2.13612	2.15056	2.24741	2.41603	2.46464
	0.8	1.99049	1.99203	2.00518	2.09354	2.24887	2.29437
	1.0	1.88482	1.88626	1.89854	1.98111	2.12713	2.17032
2	0.0	4.77723	4.77723	4.77725	4.77767	5.15761	5.19491
	0.2	4.30778	4.31047	4.33449	4.55465	4.77619	4.77652
	0.4	4.25334	4.25648	4.28440	4.52994	4.77619	4.77651
	0.6	4.23685	4.24014	4.26929	4.52257	4.77619	4.77651
	0.8	4.22891	4.23227	4.26202	4.51903	4.77619	4.77651
	1.0	4.22425	4.22764	4.25775	4.51695	4.77619	4.77651
3	0.0	5.24842	5.24852	5.24946	5.26088	5.76651	6.45561
	0.2	4.77789	4.77790	4.77796	4.77909	5.18382	5.21463
	0.4	4.77781	4.77781	4.77787	4.77887	5.18196	5.21324
	0.6	4.77778	4.77779	4.77784	4.77881	5.18139	5.21281
	0.8	4.77777	4.77777	4.77783	4.77879	5.18112	5.21261
	1.0	4.77776	4.77777	4.77782	4.77877	5.18096	5.21249
4	0.0	8.09191	8.09283	8.10115	8.18491	8.98880	10.98872
	0.2	5.26036	5.26045	5.26128	5.27141	5.79654	6.78107
	0.4	5.25952	5.25961	5.26044	5.27067	5.79514	6.77156
	0.6	5.25926	5.25936	5.26019	5.27044	5.79468	6.76850
	0.8	5.25914	5.25923	5.26007	5.27033	5.79445	6.76698
	1.0	5.25907	5.25916	5.26000	5.27027	5.79432	6.76606
5	0.0	11.08776	11.08777	11.08783	11.08840	11.09519	11.52128
	0.2	8.91371	8.91434	8.92005	8.97661	9.49709	11.14646
	0.4	8.90768	8.90832	8.91403	8.97103	9.49447	11.14622
	0.6	8.90567	8.90632	8.91204	8.96918	9.49361	11.14614
	0.8	8.90467	8.90532	8.91104	8.96825	9.49317	11.14610
	1.0	8.90407	8.90471	8.91047	8.96769	9.49292	11.14607

Table 5. Tables of Green's functions

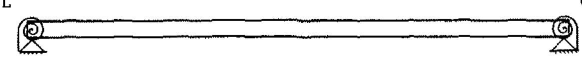
Subscripts R and L represent right ($x = L$) and left ($x = 0$) boundary attachments, respectively. Subscript P refers to diametral moment of inertia. Superscripts R and T denote rotational and translational springs

$$Q_L = k_L^R L/EI,\ Q_R = k_R^R L/EI,\ K_L = k_L^T L^3/EI,\ K_R = k_R^T L^3/EI,$$
$$M_R = m_R/mL,\ M_L = m_L/mL,\ J_R = I_{PR}/mL^3,\ J_L = I_{PL}/mL^3.$$

Where used

$$\alpha = \frac{Q_L}{2z},\quad \beta = \frac{Q_R}{z},\quad \gamma = \frac{K_R}{z^3} - zM_R,\quad \xi = \frac{Q_L}{z} - z^3 J_L,\quad \zeta = -\left(\frac{K_L}{z^3} - zM_L\right),$$
$$\lambda = \frac{Q_R}{z} - z^3 J_R,\quad \eta = \frac{1+\zeta\xi}{1-\zeta\xi},\quad \mu = \frac{2\xi}{1-\zeta\xi},\quad \nu = \frac{2\zeta}{1-\zeta\xi}.$$

Table 5(a). Simply-supported beam

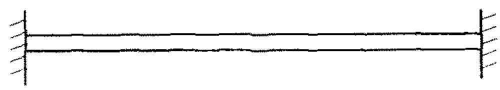


$$[e] = \begin{bmatrix} s-\alpha(c-C) & S-\alpha(c-C) \\ -s+\beta c+\alpha(\beta(s+S)+c+C) & S+\beta C+\alpha(\beta(s+S)+c+C) \end{bmatrix},$$

$$[\phi] = \begin{bmatrix} \tilde{s}-\alpha(\tilde{c}-\tilde{C}) & \tilde{c}+\alpha(\tilde{s}+\tilde{S}) \\ \tilde{S}-\alpha(\tilde{c}-\tilde{C}) & \tilde{C}+\alpha(\tilde{s}+\tilde{S}) \end{bmatrix},$$

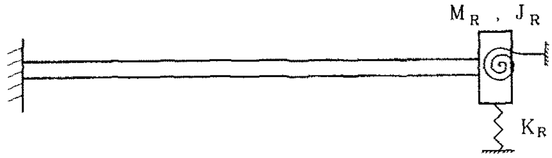
$$[\varphi] = \begin{bmatrix} \hat{s}-\hat{S} & -\hat{c}+\hat{C} \\ -(\hat{s}+\hat{S})+\beta(\hat{c}-\hat{C}) & \hat{c}+\hat{C}+\beta(\hat{s}+\hat{S}) \end{bmatrix}.$$

Table 5(b). Fixed–fixed beam



$$[e] = \begin{bmatrix} c-C & s-S \\ -s-S & c-C \end{bmatrix},\quad [\phi] = \begin{bmatrix} \tilde{c}-\tilde{C} & -\tilde{s}-\tilde{S} \\ \tilde{s}-\tilde{S} & \tilde{c}-\tilde{C} \end{bmatrix},\quad [\varphi] = \begin{bmatrix} \hat{s}-\hat{S} & -\hat{c}+\hat{C} \\ \hat{c}-\hat{C} & \hat{s}+\hat{S} \end{bmatrix}.$$

Table 5(c). Cantilever with end restraints and load



$$[e] = \begin{bmatrix} -c-C-\lambda(s+S) & -s-S+\lambda(c-C) \\ s-S-\gamma(c-C) & -c-C-\gamma(s-S) \end{bmatrix},$$

$[\phi]$ (same as in 5(b)),

$$[\varphi] = \begin{bmatrix} -\hat{s}-\hat{S}+\lambda(\hat{c}-\hat{C}) & \hat{c}+\hat{C}+\lambda(\hat{s}+\hat{S}) \\ -\hat{c}-\hat{C}+\gamma(-\hat{s}+\hat{S}) & -\hat{s}+\hat{S}+\gamma(\hat{c}-\hat{C}) \end{bmatrix}.$$

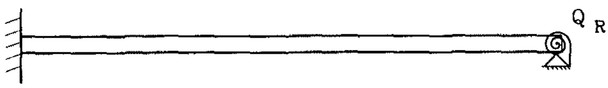
Table 5(d). Pin-free with end restraints and load

$$[e] = \begin{bmatrix} -s+\lambda c+\alpha(c+C+\lambda(s+S)) & S+\lambda C+\alpha(c+C+\lambda(s+S)) \\ -c-\gamma s+\alpha(-s+S+\gamma(c-C)) & C-\gamma S+\alpha(-s+S+\gamma(c-C)) \end{bmatrix},$$

$[\phi]$ (same as in 5(a)),

$[\varphi]$ (same as in 5(c)).

Table 5(e). Propped cantilever with end restraint

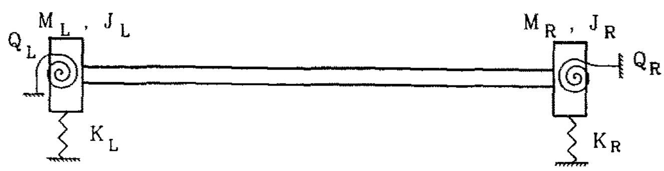


$$[e] = \begin{bmatrix} c-C & s-S \\ -c-C-\beta(s+S) & -s-S+\beta(c-C) \end{bmatrix},$$

$[\phi]$ (same as in 5(b)),

$[\varphi]$ (same as in 5(a)).

Table 5(f). Elastically supported beam with end restraint and load



$$[e] = \begin{bmatrix} -c-\lambda s+\eta(C+\lambda S)+v(S+\lambda C) & -s+\lambda c+\mu(C+\lambda S)+\eta(S+\lambda C) \\ s-\gamma c+\eta(S-\gamma C)+v(C-\gamma S) & -c-\gamma s+\mu(S-\gamma C)+\eta(C-\gamma S) \end{bmatrix},$$

$$[\phi] = \begin{bmatrix} \tilde{c}+\eta\tilde{C}+v\tilde{S} & -\tilde{s}+\eta\tilde{S}+v\tilde{C} \\ \tilde{s}+\mu\tilde{C}+\eta\tilde{S} & \tilde{c}+\mu\tilde{S}+\eta\tilde{C} \end{bmatrix},$$

$[\varphi]$ (same as in 5(c)).

8. DISCUSSION

An exact method for determining the dynamic characteristics of Euler–Bernoulli beams with attached masses and springs is given, using Green's functions. These functions have been tabulated for beams of several common boundary conditions. Some example problems with known solutions are considered, and the results confirm the correctness of the method. Moreover, the method accommodates any number of spring or mass attachments, the final result being the evaluation of a determinant (=0) whose elements are determined from the tabulated Green's functions. The method is also applicable to multi-span beams, and to the important class of periodic structures such as coupled bladed disk assemblies of a turbine shaft.

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